# Tensor product and direct sum

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From discussions with Pieter Kok (6-11 June 2007)

## 1 Tensor product

If you have a "big system", made of two or more subsystems, the hilbert space of the big system can be written as *tensor product* of smaller Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \otimes \dots \tag{1}$$

For example, if I have two systems, each of which live in an Hilbert space of dimension 2, i.e with a basis of two state vectors:

$$\mathcal{H}_a = \{|0\rangle_a, |1\rangle_a\} \mathcal{H}_b = \{|0\rangle_b, |1\rangle_b\} \tag{2}$$

then the Hilbert space describing both systems (total system) is the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$, and the basis of this total Hilbert space is the tensor product of the single bases. Since it is a product, the "cross terms" will appear:

$$\begin{aligned} &\{|0\rangle_a, |1\rangle_a\} \otimes \{|0\rangle_b, |1\rangle_b\} \\ =&\{|0\rangle_a \otimes |0\rangle_b, ...\} \\ =&\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \end{aligned} \tag{3}$$

### 1.1 dimensions

The dimension of the (space obtained as) tensor product is the product of the dimensions of the "factors": if $\mathcal{H}_a$ has dimension $n$ and $\mathcal{H}_b$ has dimension $m$, the dimension $d$ of $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_b$ is $d = n \times m$.

## 2 Direct sum

In general a matrix acts on basis elements, a turn them in other elements. So we can put a "label" on the left, indicating "on which element of the basis that row acts".

$$\left(\begin{array}{cc|cc} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ \hline m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{array}\right) \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \tag{4}$$

A generic vectors will be the sum of each basis elements.

If we have a block diagonal matrix, we see that each block acts on only a subspace. The subspaces are never mixed. This means that, for what matters to this matrix, there are two separate parts of the Hilbert space. We can separate those two parts, and to "reconstruct" the whole space, we have to do a *direct sum*

$$\left(\begin{array}{cc|cc} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ \hline 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{array}\right) \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} = M_{12} \oplus M_{34} \tag{5}$$

### 2.1 Spaces

If we introduce an Hilbert space which is the direct sum of two other Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_\alpha \oplus \mathcal{H}_\beta$$

this means that each element of $\mathcal{H}$ can be written as a linear combination of two elements, one in $\mathcal{H}_\alpha$ and one in $\mathcal{H}_\beta$:

$$\forall\ |\psi_\alpha\rangle \in \mathcal{H}_\alpha, |\psi_\beta\rangle \in \mathcal{H}_\beta; \qquad \mathcal{H} = \text{span}\{\alpha|\psi_\alpha\rangle + \beta|\psi_\beta\rangle\}_{\alpha,\beta\in\mathbb{C}} \tag{6}$$

This is the same (at finite dimensions) as the two dimensional plane being spanned by two single dimension subspaces:

$$\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R} \tag{7}$$

Is worth noting that $|i, j\rangle \neq |i, 0\rangle + |0, j\rangle$, so $\mathcal{F}^{ab} = \mathcal{E}^{ab}_{0\tilde{1}} \oplus \mathcal{E}^{ab}_{\tilde{1}0}$

## 3 n-particles Hilbert space

$$\mathcal{H}_n = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}}_{\text{n times}} \tag{8}$$

an element of this space is a state with a known number of particles (photons), in a known mode.

## 4 single mode Fock space

Several "n-particles hilbert spaces" (previous sections) form a Fock space, where the number of particles is undetermined (creation and annihilation):

$$\begin{aligned} \mathcal{F}_a &= \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \\ &= \bigoplus_{n=0}^{\infty} H_n \end{aligned} \tag{9}$$

# 5 multi-mode Fock space

$$
\begin{array}{rcl}
\mathcal{F}^m & & \\
= & & \\
\mathcal{F}_a & = & \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \\
\otimes & & \\
\mathcal{F}_b & = & \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \\
\otimes & & \\
\mathcal{F}_c & = & \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \\
\otimes & & \\
\vdots & &
\end{array}
$$

in more compact notation:

$$
\begin{aligned}
\mathcal{F}^m &= \bigotimes_i \mathcal{F}_i \qquad \text{(tensor prod. of single mode Fock sp.)} \\
&= \bigotimes_i \left[ \overbrace{\mathcal{H}^{\otimes 0} \oplus \mathcal{H}^{\otimes 1} \oplus \mathcal{H}^{\otimes 2} \oplus \ldots \oplus \mathcal{H}^{\otimes n} \oplus \ldots}^{\mathcal{F}_i} \right]_i \\
&= \bigotimes_i \left[ \overbrace{\bigoplus_n \mathcal{H}^{\otimes n}}^{\mathcal{F}_i} \right]_i \\
&= \bigotimes_i \left[ \bigoplus_n \overbrace{\left( \bigotimes_{j=0}^{n} \mathcal{H}_j \right)}^{\mathcal{H}^{\otimes n}} \right]_i
\end{aligned}
$$